Note: The actual exam will be shorter.

MAC 1114 EXAM 3 SPRING-2016

$\boxed{1}$ (15 points) Determine the amplitude, period, and phase shift of $y = 4\sin(2x - \frac{2\pi}{3})$. Then graph the function (you should graph the function for more than one period).

$y = A\sin(Bx - c)$

$A = 4, \quad B = 2, \quad c = \frac{2\pi}{3}$

Amplitude: $|A| = |4| = 4$

Period: $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi \implies$ quarter of the period $\frac{\pi}{4}$.

Phase shift: $\frac{c}{B} = \frac{\frac{2\pi}{3}}{2} = \frac{\cancel{2}\pi}{3} \cdot \frac{1}{\cancel{2}} = \frac{\pi}{3}$

So, the key points are as follows:

$x_1 = \frac{\pi}{3} \xrightarrow{\text{phase shift}} y_1 = 4 \cdot \sin(\frac{2\pi}{3} - \frac{2\pi}{3}) = 4 \cdot \sin(0) = 0$

$x_2 = \underset{x_1}{\frac{\pi}{3}} + \underset{\text{quarter period}}{\frac{\pi}{4}} = \frac{7\pi}{12} \implies y_2 = 4 \cdot \sin(\frac{14\pi}{12} - \underset{(4)}{\frac{2\pi}{3}}) = 4 \cdot \sin(\frac{\cancel{6}\pi}{\cancel{12}_2}) = 4$

$x_3 = \frac{7\pi}{12} + \underset{(3)}{\frac{\pi}{4}} = \frac{7\pi}{12} + \frac{3\pi}{12} = \frac{10\pi}{12} \implies y_3 = 4 \cdot \sin(\underbrace{\frac{20\pi}{12} - \frac{8\pi}{12}}_{=\pi}) = 0$

$x_4 = \frac{10\pi}{12} + \underset{(3)}{\frac{\pi}{4}} = \frac{10\pi}{12} + \frac{3\pi}{12} = \frac{13\pi}{12} \implies y_4 = 4 \cdot \sin(\underbrace{\frac{26\pi}{12} - \frac{8\pi}{12}}_{=\frac{3\pi}{2}}) = -4$

$x_5 = \frac{13\pi}{12} + \underset{(3)}{\frac{\pi}{4}} = \frac{13\pi}{12} + \frac{3\pi}{12} = \frac{16\pi}{12} \implies y_5 = 4 \cdot \sin(\underbrace{\frac{32\pi}{12} - \frac{8\pi}{12}}_{=2\pi}) = 0$

The pairs:

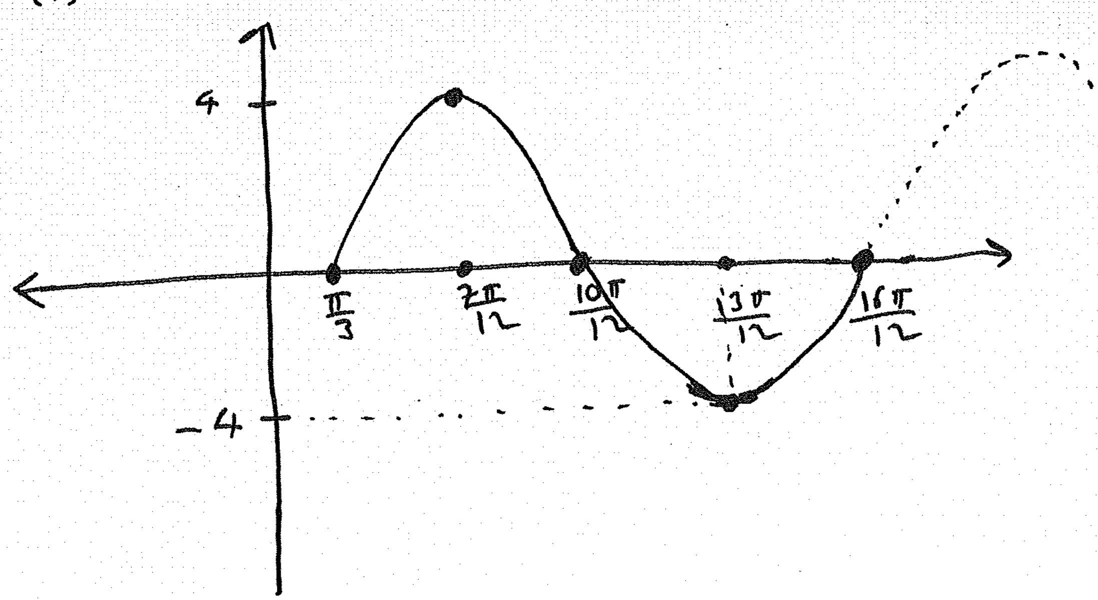
$(\frac{\pi}{3}, 0)$

$(\frac{7\pi}{12}, 4)$

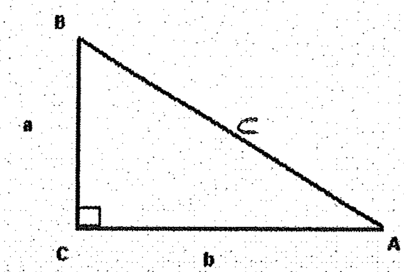
$(\frac{10\pi}{12}, 0)$

$(\frac{13\pi}{12}, -4)$

$(\frac{16\pi}{12}, 0)$



2 (10 points) Use the right triangle shown in the picture to find $a, c,$ and A We know that $b = 5$, $B = 60°$. You need to use trigonometric functions for this question, other methods will be disregarded.



One way to do it:

- $\sin(60°) = \frac{b}{c}$

$\frac{\sqrt{3}}{2} = \frac{5}{c} \Rightarrow c = \frac{10}{\sqrt{3}} = \boxed{\frac{10\sqrt{3}}{3}}$

$A + B = 90°$

$A + 60° = 90°$

$\boxed{A = 30°}$

- $\cos(60°) = \frac{a}{c}$

$\frac{1}{2} = \frac{a}{\frac{10\sqrt{3}}{3}} \Rightarrow \boxed{a = \frac{10\sqrt{3}}{6}}$

3 (20 points)

(a) (3 points) Find the exact value of $\cos^{-1}(-\frac{\sqrt{2}}{2})$. Explain your answer.

Let $\cos^{-1}(-\frac{\sqrt{2}}{2}) = \theta$. So, we are looking for an angle θ in $[0, \pi]$ such that $\cos\theta = -\frac{\sqrt{2}}{2}$.
θ has to be in quadrant II as cos is negative.

So, θ is $135° = \frac{3\pi}{4}$ because $\cos(135°) = -\cos(45°) = -\frac{\sqrt{2}}{2}$

Thus, $\cos^{-1}(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$.

(b) (3 points) Find the exact value of $\sin(\underbrace{\sin^{-1}(-2\pi)}_{\text{let } = \theta})$.

If $\sin^{-1}(-2\pi) = \theta$, that means $\sin(\theta) = -2\pi \approx -6.28$
but there is no angle θ such that $\sin(\theta) \approx -6.28$
because the range of sine is $[-1, 1]$.
So, the answer is undefined.

(c) (4 points) Find the exact value of $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$. Explain your answer.

Since $\frac{5\pi}{6}$ is not in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the answer is NOT $\frac{5\pi}{6}$.

$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ by the reference angle idea.

$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \sin^{-1}\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$ because $\frac{\pi}{6}$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(d) (5 points) Find the exact value of $\tan\left(\sin^{-1}\left(-\frac{4}{5}\right)\right)$.

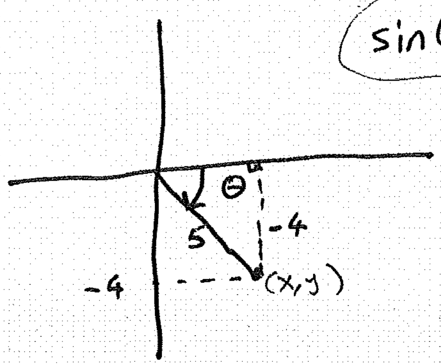
Let $\theta = \sin^{-1}\left(-\frac{4}{5}\right)$, so we are looking for $\tan(\theta)$?!

$\downarrow$

this means $\sin(\theta) = -\frac{4}{5}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

but sine is negative θ has to be in $\left[-\frac{\pi}{2}, 0\right]$ interval.

$\sin(\theta) = \frac{y}{r} = \frac{-4}{5}$



Therefore, by Pythogeran theorem $x = 3$

$\Rightarrow$ and, so $\tan(\theta) = \frac{y}{x} = \frac{-4}{3} = -\frac{4}{3}$

$\Rightarrow$ $\tan\left(\sin^{-1}\left(-\frac{4}{5}\right)\right) = -\frac{4}{3}$

(e) (5 points) Find the exact value of $\sin(75°)$

$\sin(75°) = \sin(30° + 45°)$ by sum formula for sine

$= \sin(30°) \cdot \cos(45°) + \sin(45°) \cdot \cos(30°)$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4}$.

$\boxed{4}$ (15 points)

(a) (5 points) Verify the identity

$$\cos\theta \cdot \csc\theta \cdot \tan\theta = 1$$

The left hand side is

$$= \cos(\theta) \cdot \frac{1}{\cancel{\sin(\theta)}} \cdot \frac{\cancel{\sin(\theta)}}{\cos(\theta)}$$

$$= \cancel{\cos(\theta)} \cdot \frac{1}{\cancel{\cos(\theta)}} = 1 \checkmark$$

(b) (10 points) Verify the identity

$$\frac{\overset{\frac{1}{\cos(2\theta)}}{\sec(2\theta)} - \cos(2\theta)}{\sin^2(2\theta)} = \sec(2\theta)$$

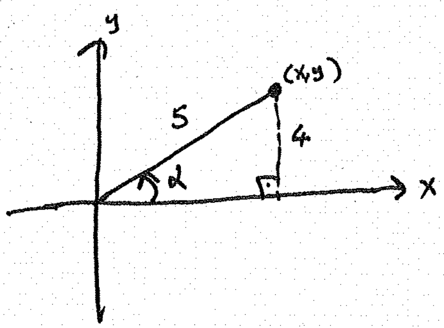
From the left hand side:

$$= \frac{\frac{1}{\cos(2\theta)} - \cos(2\theta)}{\sin^2(2\theta)} = \frac{\frac{1-\cos^2(2\theta)}{\cos(2\theta)}}{\sin^2(2\theta)}$$

$$= \frac{\cancel{(1-\cos^2(2\theta))}}{\cos(2\theta)} \cdot \frac{1}{\cancel{\sin^2(2\theta)}} = \frac{1}{\cos(2\theta)}$$

$$= \sec(2\theta)$$ because $\sin^2(2\theta) = 1 - \cos^2(2\theta)$ by Pythagorean.

5 (15 points) Given that $\sin(\alpha) = \frac{4}{5}$, α lies in quadrant I, and $\sin(\beta) = \frac{4}{5}$, β lies in quadrant II. Find the exact value of $\sin(\alpha - \beta)$.



$\sin(\alpha) = \frac{y}{r} = \frac{4}{5}$

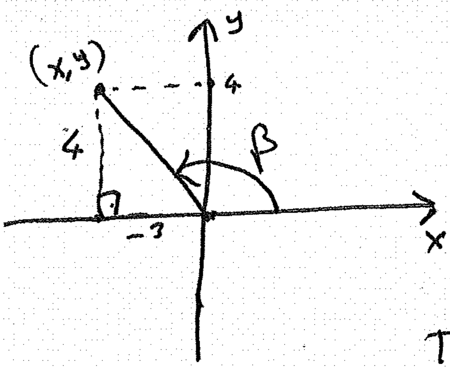
By Pythagorean Identity, $x^2+y^2=r^2$

$x^2+4^2=5^2$

$x^2=9$

$x=3$

$\Rightarrow \cos(\alpha) = \frac{x}{r} = \frac{3}{5}$



$\sin(\beta) = \frac{y}{r} = \frac{4}{5}$, again by Pythagorean Identity

$x^2+y^2=r^2$

$x^2 = 5^2-4^2=9$

$x=-3$

$\Rightarrow \cos(\beta) = \frac{x}{r} = -\frac{3}{5}$

Therefore, $\sin(\alpha-\beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$

$= \frac{4}{5}\cdot\left(-\frac{3}{5}\right) - \frac{3}{5}\cdot\frac{4}{5}$

$= \frac{-12}{25} - \frac{12}{25} = \boxed{-\frac{24}{25}}$

6 (10 points) Verify the identity

$$(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$$

Let's start from the left hand side:

$(\sin\theta - \cos\theta)^2 = \sin^2\theta - 2\sin\theta\cdot\cos\theta + \cos^2\theta$

$= \underbrace{\sin^2\theta + \cos^2\theta}_{=1} - \underbrace{2\cdot\sin\theta\cdot\cos\theta}_{\sin(2\theta)}$

$= 1 - \sin(2\theta)$ ✓

by the double angle formula.

MAC 1114 EXAM 3

$\boxed{7}$ (20 points)

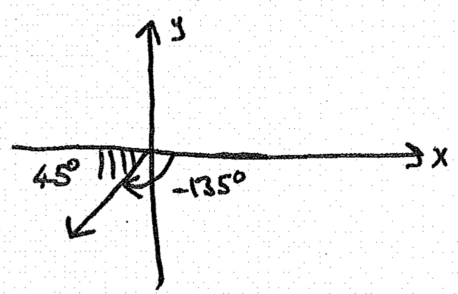
(a) (10 points) Find the value of $2 + \sin^2(75°) + \sin^2(15°)$. Explain your answer.

$= 2 + \sin^2(75°) + \cos^2(75)$ cofunctions

$= 2 + 1$ by Pythagorean Identity

$= 3$

(b) (10 points) Use the reference angle to find the exact value of $\sin(-135°)$. Explain your answer.



The refrence angle of $-135°$ is $45°$.

So, $\sin(-135°) = \mp \sin(45°)$.

But, we choose the negative sign because sin is negative in the third quadrant.

$\sin(-135°) = -\sin(45°) = -\frac{\sqrt{2}}{2}$

$\boxed{8}$ (Bonus 5 points) Find the exact

$$\cos^2(22.5°) - \sin^2(22.5°)$$

$= \cos(2 \cdot (22.5°))$

$= \cos(45°) = \frac{\sqrt{2}}{2}$

because of the double angle formula:

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

EXAM THREE

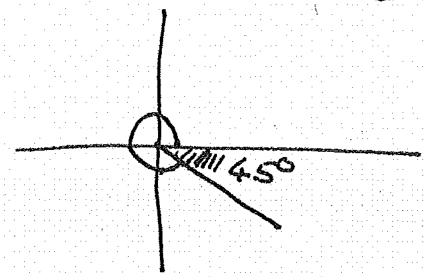
$\boxed{9}$ (25 points)

(a) (10 points) Solve the equation over the interval $[0, 2\pi)$

$$\cos(2x) = \frac{\sqrt{2}}{2}$$

We know $\cos(45^\circ) = \frac{\sqrt{2}}{2}$ and $\cos(315^\circ) = \frac{\sqrt{2}}{2}$

$\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$



Since $\cos(2x) = \frac{\sqrt{2}}{2}$

$2x = \frac{\pi}{4} + 2n\pi$ since cos has period 2π.

$2x = \frac{7\pi}{4} + 2n\pi$

If $n=0$, $x = \frac{\pi}{8}$ and $x = \frac{7\pi}{8}$

$n=1$, $x = \frac{\pi}{8} + \pi = \frac{9\pi}{8}$, and $x = \frac{7\pi}{8} + \pi = \frac{15\pi}{8}$

If $n=2$, we are not in the range $[0, 2\pi)$. So, the sol. is $x = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$.

(b) (15 points) Solve the following equation on the interval $[0, 2\pi)$

$$\sin x \cdot \cos x = -\frac{\sqrt{3}}{4}$$

Remember that $\sin 2x = 2\sin x \cdot \cos x$

$\Rightarrow 2 \cdot \sin x \cdot \cos x = -\frac{\sqrt{3}}{2}$

$\Rightarrow \sin 2x = -\frac{\sqrt{3}}{2}$, we know $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$.

$\Rightarrow 2x = \frac{4\pi}{3} + 2n\pi$

and $2x = \frac{5\pi}{3} + 2n\pi$

If $n=0$, $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{6}$

If $n=1$, $x = \frac{5\pi}{3}$, and $x = \frac{11\pi}{6}$.

The sol. set is $\left\{ \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6} \right.$